IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Variational methods for periodic orbits of reduced Hamiltonian systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 275212

(http://iopscience.iop.org/1751-8121/41/27/275212)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.149
The article was downloaded on 03/06/2010 at 06:58

Please note that terms and conditions apply.

J. Phys. A: Math. Theor. **41** (2008) 275212 (20pp) doi:10.1088/1751-8113/41/27/275212

Variational methods for periodic orbits of reduced Hamiltonian systems

Yoshiro Yabu

Department of Applied Mathematics and Physics, Kyoto University, Kyoto-606-8501, Japan

E-mail: yoshiro@amp.i.kyoto-u.ac.jp

Received 17 December 2007, in final form 10 May 2008
Published 16 June 2008
Online at stacks.iop.org/JPhysA/41/275212

Abstract

A variational method for periodic orbits is not easy to apply to a Hamiltonian system, when the symplectic form is not exact. However, if the Hamiltonian system in question is a reduced one from a Hamiltonian system on an exact symplectic manifold, the variational method applies to the latter system in order to find periodic orbits of the reduced system. This paper studies variational methods for periodic orbits in the systems reduced by the Marsden–Weinstein and the orbit reduction procedures. Periodic orbits of the reduced systems are characterized as critical points of action functionals for loops in the original phase space together with Lagrange multipliers.

PACS numbers: 45.10.Db, 45.20.Jj

1. Introduction

In classical mechanics, the variational principle provides a method for obtaining the Hamilton equations. In particular, the variational method is applied to the action functional on the loop space to find periodic orbits as critical points of the functional. Let P be a smooth manifold, and T^*P its cotangent bundle equipped with the Liouville one-form θ together with the canonical symplectic structure $\omega = -\mathrm{d}\theta$. We define the action functional for one-periodic loops $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to T^*P$ to be

$$\mathcal{A}_H(\gamma) = \int_{S^1} \gamma^*\theta - \int_0^1 H(t, \gamma(t))\,\mathrm{d}t, \tag{1.1}$$

where H is a time-periodic Hamiltonian function. Then critical points of this functional give one-periodic orbits of the Hamilton equations

$$\dot{q} = \frac{\partial H}{\partial p}(t, q, p), \qquad \dot{p} = -\frac{\partial H}{\partial q}(t, q, p),$$

which are written in the canonical coordinates (q, p) of the cotangent bundle. This method also works well if the phase spaces are manifolds endowed with exact symplectic forms.

1751-8113/08/275212+20$30.00
© 2008 IOP Publishing Ltd Printed in the UK

On the other hand, symmetry of Hamiltonian systems is of great interest. In particular, there are two major procedures to reduce Hamiltonian systems with the symmetry; one is the Marsden–Weinstein reduction, and the other the orbit reduction. By means of symmetry, the system can be reduced to that on the reduced phase space, but the variational method for periodic orbits of the reduced system cannot be written down for loops in the reduced space. This is because the reduced symplectic form is not always an exact form, and we cannot define the action functional on the loop space of the reduced phase space like (1.1).

Some approaches to variational methods [CM87, CIM87, IO96] have been attempted for reduced dynamical systems. In the case of exact symplectic manifolds, Cendra and Marsden [CM87] have considered the action functional on the space of paths with fixed end points in the original phase space. They have shown that the functional induces one on the space of paths with fixed end points in the reduced phase space, and that critical points of the induced functional are paths which are subject to the reduced equation. Papers [CM87, CIM87] have also studied variational methods for reduced Lagrangian systems in terms of Clebsch variables and Lin constraints. Ibort and Ontalba [IO96] have considered the action functional $\mathcal{A}_H$ restricted to the free loop space of a level set of the momentum map, and shown that critical points of the restricted functional projects to periodic orbits of the reduced system.

This paper develops the idea of [IO96] to formulate variational methods for the reduced systems by both of the Marsden–Weinstein and orbit reductions. However, instead of the restriction of the action functional to a level set of the momentum map, the action functional is dealt with on the free loop space of the original phase space along with time-dependent Lagrange multipliers. Throughout this paper, the unreduced symplectic manifolds are assumed to be exact.

This paper is organized as follows: in section 2, a brief review is made of momentum maps, Marsden–Weinstein and orbit reductions, and relative periodic orbits of the Hamiltonian system. Further, a variational method is given for relative periodic orbits. In sections 3 and 4, variational methods are formulated for periodic orbits of the systems reduced by the Marsden–Weinstein and by the orbit reductions, respectively. Both methods are shown to yield the same result.

2. Symplectic reductions

We make a brief review of momentum maps, the Marsden–Weinstein reduction and the orbit reduction. See [OR04] for the detail of the reduction theory. We also formulate a variational method for finding relative periodic orbits of the Hamilton equation in the tail of this section.

Let (M, Ω) be an exact symplectic manifold, where the symplectic form Ω on M is an exact two-form. We assume that a connected Lie group G properly and symplectically acts on M from the left,

$$\Psi : G \times M \longrightarrow M; (g, x) \longmapsto \Psi_g(x) = gx.$$

In addition, we impose the condition that there is a one-form Θ which satisfies $\Omega = -\mathrm{d}\Theta$ and is invariant under the G-action; $g^*\Theta = \Psi_g^*\Theta = \Theta$ for any $g \in G$. In this case, the action of G on M admits a momentum map $\mu : M \to \mathfrak{g}^* = \mathrm{Lie}(G)^*$ defined through

$$(\mu(x), \xi) = \Theta_x(\xi_M), \qquad x \in M, \xi \in \mathfrak{g} := \mathrm{Lie}(G), \tag{2.1}$$

where $(\bullet, \bullet)$ is the natural paring between $\mathfrak{g}^*$ and $\mathfrak{g}$, and ξ_M is the fundamental vector field on M induced by $\xi \in \mathfrak{g}$. Further, the function $\mu : M \to \mathfrak{g}^*$ is G-equivariant, and satisfies $(\mathrm{d}\mu(X), \xi) = \Omega(\xi_M, X)$ for an arbitrary vector field X on M and $\xi \in \mathfrak{g}$. This implies that ξ_M is a Hamiltonian vector field associated with the function $\Theta(\xi_M) : M \to \mathbb{R}$. For this reason, the G-action admitting a G-equivariant momentum map is said to be Hamiltonian.

To fix the sign convention, we summarize here the Hamiltonian vector fields, coadjoint action and G-equivariance.

(1) The Hamiltonian vector field X_H associated with a smooth function $H \in C^\infty(M)$ is uniquely determined through

$$i_{X_H}\Omega = \mathrm{d}H, \tag{2.2}$$

where $i_{X_H}\Omega$ denotes the interior product of X_H and Ω.

(2) The coadjoint action of G on $\mathfrak{g}^*$ is defined through

$$(\mathrm{Ad}^*_g\lambda, \xi) = (\lambda, \mathrm{Ad}_{g^{-1}}\xi)$$

for any $\lambda \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$ and $g \in G$. This means that Ad^*_g is the dual not to Ad_g but to $\mathrm{Ad}_{g^{-1}}$. Thus, ad^* is a derived representation of the Ad^*-action of G; $\mathrm{ad}^*_\xi = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\mathrm{Ad}^*_{\exp(t\xi)}$.

(3) The G-equivariance of the momentum map $\mu : M \to \mathfrak{g}^*$ is expressed as

$$\mu(gx) = \mathrm{Ad}^*_g\mu(x), \qquad g \in G, x \in M.$$

Indeed, the momentum map μ defined by (2.1) satisfies

$$(\mu(gx), \xi) = \Theta_{gx}(\xi_M) = (\Phi^*_{g^{-1}}\Theta_x)(\xi_M) = \Theta_x((\mathrm{Ad}_{g^{-1}}\xi)_M) = (\mu(x), \mathrm{Ad}_{g^{-1}}\xi)$$

for any $g \in G$, $x \in M$ and $\xi \in \mathfrak{g}$.

We proceed to a review of reduction procedure. Let $\lambda \in \mathfrak{g}^*$ be a regular value of the momentum map μ. Denote by G_λ the isotropy subgroup of G at λ by the Ad^*-action on $\mathfrak{g}^*$. If G_λ freely acts on the regular level set $\mu^{-1}(\lambda) \subset M$, then the quotient space

$$M_\lambda := \mu^{-1}(\lambda)/G_\lambda$$

becomes a smooth manifold, and is endowed with a symplectic form Ω_λ determined through

$$\pi^*_\lambda\Omega_\lambda = \Omega|_{\mu^{-1}(\lambda)}, \tag{2.3}$$

where $\pi_\lambda : \mu^{-1}(\lambda) \to M_\lambda$ is the natural projection. We note here that such a symplectic structure Ω_λ is unique. In particular, we write the reduced phase space as $M /\!\!/ G = (M_0, \Omega_0)$ when $\lambda = 0$. The procedure for the construction of the symplectic manifold $(M_\lambda, \Omega_\lambda)$ is called the Marsden–Weinstein reduction [MW74]. Suppose a time-dependent Hamiltonian $H \in C^\infty(S^1 \times M)$ of period one in t, which satisfies

$$H(t, gx) = H(t, x) \qquad \text{for } (t, x) \in S^1 \times M \text{ and } g \in G,$$

where S^1 denotes $\mathbb{R}/\mathbb{Z}$ throughout this paper otherwise mentioned. Then there is a smooth function $H^\lambda \in C^\infty(S^1 \times M_\lambda)$ such that $\pi^*_\lambda H^\lambda_t = H_t|_{\mu^{-1}(\lambda)}$, where H_t and H^λ_t denote the functions $H(t, \bullet)$ and $H^\lambda(t, \bullet)$, respectively. Thus, the original Hamiltonian system (M, Ω, H) with the Hamilton equation

$$\dot{x} = X_{H_t}(x) \tag{2.4}$$

is reduced to the Hamiltonian system $(M_\lambda, \Omega_\lambda, H^\lambda)$ with the reduced Hamilton equation

$$\dot{y} = X_{H^\lambda_t}(y). \tag{2.5}$$

We call this reduced system the Marsden–Weinstein reduced system.

In comparison with the Marsden–Weinstein reduction, the orbit reduction [KKS78] runs as follows: let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^*$ through a regular value λ of μ. Suppose that the action of G on $\mu^{-1}(\mathcal{O})$ is free. This condition is equivalent to that G_λ freely acts on $\mu^{-1}(\lambda)$. Since $\mu^{-1}(\mathcal{O})$ is a smooth manifold and admits a free action of G, the quotient space

$$M_\mathcal{O} := \mu^{-1}(\mathcal{O})/G$$

has a smooth manifold structure. A unique symplectic form $\Omega_{\mathcal{O}}$ on $M_{\mathcal{O}}$ is defined through

$$\pi_{\mathcal{O}}^*\Omega_{\mathcal{O}} = \Omega|_{\mu^{-1}(\mathcal{O})} - (\mu|_{\mu^{-1}(\mathcal{O})})^*\omega_{\mathcal{O}}, \tag{2.6}$$

where $\pi_{\mathcal{O}} : \mu^{-1}(\mathcal{O}) \to M_{\mathcal{O}}$ is the canonical projection, and $\omega_{\mathcal{O}}$ is the Kirillov–Kostant–Souriau (KKS) form on the coadjoint orbit $\mathcal{O}$ defined to be

$$\omega_{\mathcal{O}}(\mathrm{ad}_\xi^*\lambda, \mathrm{ad}_\eta^*\lambda) := (\lambda, [\xi, \eta]), \qquad \lambda \in \mathcal{O}, \xi, \eta \in \mathfrak{g}.$$

The reduced space $(M_{\mathcal{O}}, \Omega_{\mathcal{O}})$ is symplectically diffeomorphic to $(M_\lambda, \Omega_\lambda)$. Indeed, the inclusion $\mu^{-1}(\lambda) \hookrightarrow \mu^{-1}(\mathcal{O})$ induces a symplectic diffeomorphism from $M_\lambda = \mu^{-1}(\lambda)/G_\lambda$ to $M_{\mathcal{O}} = \mu^{-1}(\mathcal{O})/G$. See [OR04] for the proof. When H_t is invariant under the G-action at each t, it projects to a smooth function $H_t^{\mathcal{O}} \in C^\infty(M_{\mathcal{O}})$ through $\pi_{\mathcal{O}}$, that is, $\pi_{\mathcal{O}}^* H_t^{\mathcal{O}} = H_t|_{\mu^{-1}(\mathcal{O})}$. Then the original Hamiltonian system (M, Ω, H) reduces to the system $(M_{\mathcal{O}}, \Omega_{\mathcal{O}}, H^{\mathcal{O}})$ with the Hamilton equation

$$\dot{z} = X_{H_t^{\mathcal{O}}}(z). \tag{2.7}$$

The system $(M_{\mathcal{O}}, \Omega_{\mathcal{O}}, H^{\mathcal{O}})$ is called the orbit reduced system.

Note here that the Hamiltonian vector field X_{H_t} is pushed forward to $X_{H_t^{\mathcal{O}}}$ accordingly,

$$(\pi_{\mathcal{O}})_* X_{H_t} = X_{H_t^{\mathcal{O}}}. \tag{2.8}$$

In fact, by using (2.2) and (2.6) along with the definition of $H_t^{\mathcal{O}}$, we obtain

$$\Omega_{\mathcal{O}}((\pi_{\mathcal{O}})_* X_{H_t} - X_{H_t^{\mathcal{O}}}, (\pi_{\mathcal{O}})_* Y) + \mu^*\omega_{\mathcal{O}}(X_{H_t}, Y) = 0,$$

for any vector field Y on M. If Y is a fundamental vector field generated by an element of $\mathfrak{g}$, $(\pi_{\mathcal{O}})_* Y$ vanishes, so that the above equation results in $\omega_{\mathcal{O}}(\mu_* X_{H_t}, \mu_* Y) = 0$. Since $\mu_* Y$ is viewed as arbitrary on $\mathcal{O}$, one has $\mu_* X_{H_t} = 0$. This implies in turn that $(\pi_{\mathcal{O}})_* X_{H_t} = X_{H_t^{\mathcal{O}}}$, as $\pi_{\mathcal{O}}$ is surjective.

Example 2.1. A typical example of the reduction is taken for the case where M is a cotangent bundle. When G smoothly acts on a manifold P, the action is naturally lifted to the cotangent bundle T^*P, on which the Liouville one-form θ is given by

$$\theta_\alpha(X) := \alpha(p_* X), \qquad \alpha \in T^*P, X \in T_\alpha(T^*P),$$

where p is the projection from T^*P to P. The lifted action of G preserves the Liouville one-form θ, so that the action of G on T^*P is symplectic with respect to the standard symplectic structure $\omega = -\mathrm{d}\theta$.

If G freely acts on P, then P is made into a principal G-bundle, $P \to P/G =: B$. In this case, the symplectic quotient $M/\!\!/G = T^*P/\!\!/G$ is symplectically diffeomorphic to T^*B. Furthermore, T^*P/G is diffeomorphic to the Whitney sum bundle of T^*B and the associated coadjoint bundle $\mathfrak{g}_P^* = P \times_{\mathrm{Ad}^*} \mathfrak{g}^*$, and has symplectic leaves $M_{\mathcal{O}} \cong T^*B \times_B \mathcal{O}_P$, where $\mathcal{O}_P = P \times_{\mathrm{Ad}^*} \mathcal{O}$. See [OR04, MP00] and papers therein for the detail of the cotangent bundle reduction. In particular, if $P = G$, then the orbit reduced space $M_{\mathcal{O}}$ is identified with the coadjoint orbit $\mathcal{O}$ equipped with the KKS form $\omega_{\mathcal{O}}$, as T^*B consists of a single point.

In what follows, we work with a time-periodic Hamiltonian system, i.e., $H_{t+1} = H_t$. Because of $H_{t+1} = H_t$, the flow ϕ^t of X_{H_t} satisfies

$$\phi^{t+1} = \phi^t \circ \phi^1 = \phi^1 \circ \phi^t.$$

If H_t is invariant under the G-action for each $t \in S^1 = \mathbb{R}/\mathbb{Z}$, then the quantity $\mu(x(t))$ is preserved along a solution $x(t)$ of (2.4). In fact, we have, for any $\xi \in \mathfrak{g}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu(x(t)), \xi) = \Omega(\xi_M, \dot{x}) = -\mathrm{d}H_t(\xi_M) = 0.$$

The last equality is due to the G-invariance of H_t.

We are now interested in periodic orbits of the Hamilton equation (2.4). If $x(t)$ is a periodic orbit of (2.4), then $x(t)$ projects to periodic orbits of the reduced Hamilton equations (2.5) and (2.7). However, for a periodic orbit $y(t)$ of (2.7), its lift on M is not expected to be a periodic orbit of (2.4), in general. The lifts are relative periodic orbits of (2.4). The definition of relative periodic orbits is as follows.

Definition 2.2. *Let $x(t)$ be a solution of the Hamilton equation (2.4). The $x(t)$ is called a relative periodic orbit of period one if there exists $g_0 \in G$ independent of t such that*

$$x(t+1) = g_0 \cdot x(t), \qquad t \in \mathbb{R}.$$

Put another way, the condition is expressed as $\phi^1(x(t)) = g_0 \cdot x(t)$.

The notion of relative periodic orbits will be useful in the successive sections, where variational methods for periodic orbits of the reduced equations (2.5) and (2.7) are described and applied.

Periodic orbits of the reduced equations are related to relative periodic orbits of the original Hamilton equation as follows: if a relative periodic orbit $x(t)$ of (2.4) lies on $\mu^{-1}(\lambda)$, then according to the Marsden–Weinstein reduction procedure, $x(t)$ projects to a loop $y(t) = \pi_\lambda(x(t))$, which becomes a periodic orbit of the Marsden–Weinstein reduced system (2.5). Conversely, let $y(t)$ be a periodic orbit of (2.5), and x_0 be a point on $\mu^{-1}(\lambda)$ such that $\pi_\lambda(x_0) = y(0)$. Then $x(t) = \phi^t(x_0)$ is a relative periodic orbit of the Hamilton equation (2.4).

In contrast with the above, we may apply the orbit reduction procedure. Let $\mathcal{O}$ be a coadjoint orbit through $\lambda \in \mathfrak{g}^*$, and $x(t)$ be a relative periodic orbit of (2.4) lying on $\mu^{-1}(\mathcal{O})$. By (2.8), the $x(t)$ projects to a periodic orbit $z(t) = \pi_{\mathcal{O}}(x(t))$ of the orbit reduced system (2.7). Conversely, let $z(t)$ be a periodic orbit of (2.7), and $x_0 \in \mu^{-1}(\mathcal{O})$ be a point such that $\pi_{\mathcal{O}}(x_0) = z(0)$. Then $x(t) = \phi^t(x_0)$ also becomes a relative periodic orbit of (2.4).

In view of these, finding periodic orbits of the reduced systems (2.5) and (2.7) amounts to finding relative periodic orbits of the original system (2.4).

Example 2.3. We take example 2.1 with restriction to $P = G$. In this case, the momentum map $\mu : T^*G \to \mathfrak{g}^*$ is put in the form

$$\mu(\alpha_g) = \mathrm{R}_g^*\alpha_g, \qquad g \in G, \alpha_g \in T_g^*G,$$

where R_g denotes the right translation by $g \in G$. The reduced space $M_{\mathcal{O}} = \mu^{-1}(\mathcal{O})/G$ is naturally identified with the coadjoint orbit $\mathcal{O}$ itself, so that the canonical projection $\pi_{\mathcal{O}} : \mu^{-1}(\mathcal{O}) \to M_{\mathcal{O}} \cong \mathcal{O}$ is regarded as the restriction of μ to $\mu^{-1}(\mathcal{O})$.

Now we assume that $\mathfrak{g}$ is endowed with an Ad-invariant inner product $\langle \bullet, \bullet \rangle$, and that there is a symmetric and positive-definite operator $I : \mathfrak{g} \to \mathfrak{g}$ with respect to this inner product. For simplicity, we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Ad-invariant inner product, and then T^*G with TG. Let H be an autonomous Hamiltonian on $T^*G \cong TG \cong G \times \mathfrak{g}$ given by

$$H(g, \xi) = \tfrac{1}{2}\langle \xi, I^{-1}(\xi) \rangle \qquad \text{for} \quad g \in G \text{ and } \xi \in \mathfrak{g}.$$

Since the reduced Hamiltonian $H^{\mathcal{O}}$ is expressed as $H^{\mathcal{O}}(\xi) = \frac{1}{2}\langle \xi, I^{-1}(\xi) \rangle$, the reduced equation is written down in the form

$$\dot{\xi} = [\xi, I^{-1}(\xi)]. \tag{2.9}$$

Thus, if ξ_0 is an eigenvector of I lying on $\mathcal{O}$, then it is a fixed point of the above reduced equation. Further, since the unreduced Hamilton equation $\dot{x} = X_H(x)$ is equivalent to the equations

$$\dot{\xi} = [\xi, I^{-1}(\xi)], \qquad g^{-1}\dot{g} = \xi,$$

a solution of $\dot{x} = X_H(x)$ passing $(g_0, \xi_0) \in G \times \mathfrak{g}$ at $t = 0$ is given by $(\exp(t\xi_0)g_0, \xi_0)$. This solution is a relative periodic orbit, and projects to a fixed point ξ_0 of the reduced equation (2.9).

The rest of this section deals with the formulation of a variational method for relative periodic orbits of the Hamilton equation (2.4). We consider the space of paths in M,

$$\mathcal{P} := \{\gamma : \mathbb{R} \to M | \gamma(t+1) = g_0\gamma(t) \text{ for some } g_0 \in G\}.$$

The tangent space to $\mathcal{P}$ at γ can be identified with

$$T_\gamma\mathcal{P} = \{X \in \Gamma(\gamma^*TM) | \exists\xi \in \mathfrak{g} \text{ s.t. } X(t+1) - (g_0)_* X(t) = \xi_M(\gamma(t+1))\},$$

where $g_0 \in G$ is determined by $\gamma(t+1) = g_0\gamma(t)$. In fact, when we take a smooth map $u : (-\epsilon, \epsilon) \times \mathbb{R} \to M$ such that

$$u(0,t) = \gamma(t) \qquad \text{and} \qquad u(s, t+1) = g(s)u(s,t) \text{ for some } g(s) \in G,$$

a one-parameter family of paths in $\mathcal{P}$ is induced by $(-\epsilon, \epsilon) \to \mathcal{P}; s \mapsto u(s, \bullet)$, which has the tangent vector $X = (\partial_s u)|_{s=0} \in T_\gamma\mathcal{P}$ at $s = 0$. Thus, by differentiating $u(s, t+1) = g(s)u(s,t)$ with respect to s at $s = 0$, we obtain $X(t+1) - g(0)_* X(t) = (\mathrm{d}g/\mathrm{d}s)g^{-1}|_{s=0}(\gamma(t+1))$ along γ.

Now we define an action functional on $\mathcal{P}$ to be

$$\mathcal{A}_H(\gamma) := \int_{[T,T+1]} \gamma^*\Theta - \int_T^{T+1} H(t, \gamma(t))\,\mathrm{d}t, \qquad \gamma \in \mathcal{P},$$

where $T \in \mathbb{R}$ is a constant. The action functional $\mathcal{A}_H$ is independent of the choice of T. In fact, since Θ and H are invariant under the G-action and since $H_{t+1} = H_t$, the integrands $\Theta_{\gamma(t)}(\dot{\gamma}(t))$ and $H(t, \gamma(t))$ are one-periodic in t, so that the action $\mathcal{A}_H(\gamma)$ for $\gamma \in \mathcal{P}$ is equal to

$$\mathcal{A}_H(\gamma) = \int_0^1 \Theta_{\gamma(t)}(\dot{\gamma})\,\mathrm{d}t - \int_0^1 H(t, \gamma(t))\,\mathrm{d}t.$$

Thus, we may put $T = 0$ without loss of generality.

We consider the variation of the action functional $\mathcal{A}_H$ at $\gamma \in \mathcal{P}$. Let $u : (-\epsilon, \epsilon) \times \mathbb{R} \to M$ be the smooth map as above and let $X = (\partial_s u)|_{s=0} \in T_\gamma\mathcal{P}$ be the variational vector field along γ. The first variational formula for the action functional $\mathcal{A}_H$ is given by

$$\begin{aligned}(\mathrm{d}\mathcal{A}_H)_\gamma(X) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{A}_H(u(s,\bullet)) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left\{ -\int_{[0,s]\times[0,1]} u^*\Omega - \int_0^1 H(t, u(s,t))\,\mathrm{d}t \right. \\ &\quad \left. + \int_0^s \Theta_{u(s',1)}\left(\frac{\partial u}{\partial s'}\right)\mathrm{d}s' - \int_0^s \Theta_{u(s',0)}\left(\frac{\partial u}{\partial s'}\right)\mathrm{d}s' \right\} \\ &= \int_0^1 \Omega(\dot{\gamma} - X_{H_t}(\gamma), X)\,\mathrm{d}t + \Theta_{\gamma(1)}(X(1)) - \Theta_{\gamma(0)}(X(0)).\end{aligned}$$

Here we have used $\Omega = -\mathrm{d}\Theta$ and the Stokes theorem in the second equality. This formula implies that $\gamma \in \mathcal{P}$ is a relative periodic orbit of (2.4) if and only if the variation $(\mathrm{d}\mathcal{A}_H)_\gamma(X)$ vanishes for any $X \in T_\gamma\mathcal{P}$ which is subject to the condition

$$\Theta_{\gamma(1)}(X(1)) - \Theta_{\gamma(0)}(X(0)) = 0.$$

Hence the variational method for relative periodic orbits of the Hamiltonian system (2.4) is phrased as follows.

Theorem 2.4. *Define a distribution $\mathcal{D}$ on $\mathcal{P}$ to be*

$$\mathcal{D}_\gamma := \{X \in T_\gamma\mathcal{P} | \Theta_{\gamma(1)}(X(1)) - \Theta_{\gamma(0)}(X(0)) = 0\}.$$

An element $\gamma \in \mathcal{P}$ is a relative periodic orbit of the Hamiltonian system (2.4) if and only if the restriction of $(\mathrm{d}\mathcal{A}_H)_\gamma$ to $\mathcal{D}_\gamma$ vanishes.

We will make a comment on the boundary condition $\Theta_{\gamma(1)}(X(1)) - \Theta_{\gamma(0)}(X(0)) = 0$ in the last section.

Since, for $X \in T_\gamma\mathcal{P}$, there exists $\xi \in \mathfrak{g}$ satisfying $X(1) - (g_0)_* X(0) = \xi_M(\gamma(1))$, the condition determining $\mathcal{D}$ is rewritten in terms of the momentum map as

$$0 = \Theta_{\gamma(1)}(X(1)) - \Theta_{\gamma(0)}(X(0)) = \Theta_{\gamma(1)}(X(1) - (g_0)_* X(0)) = (\mu(\gamma(1)), \xi).$$

If we restrict $\mathcal{P}$ to a loop space, the boundary condition $\Theta_{\gamma(1)}(X(1)) - \Theta_{\gamma(0)}(X(0)) = 0$ is satisfied automatically. The restriction to the loop space is not restrictive for dealing with periodic orbits of the reduced equations (2.5) and (2.7).

3. Variational method for the Marsden–Weinstein reduced system

This section deals with a variational method for periodic orbits of the Marsden–Weinstein reduced system (2.5). Since the reduced symplectic form Ω_λ cannot be an exact two-form, instead of formulating the variational principle on the loop space of the reduced space M_λ, we consider an action functional for loops on M together with Lagrange multipliers taking values in $\mathfrak{g}$ in order to restrict orbits on the level set $\mu^{-1}(\lambda)$. Then the reduction by G-symmetry will yield one-to-one correspondence between periodic orbits of (2.5) and critical points of the action functional up to gauge symmetry.

Throughout this section, we impose the following assumption so as to get the Marsden–Weinstein reduction work well.

Assumption 3.1. *An element $\lambda \in \mathfrak{g}^*$ is a regular value of the momentum map μ given by (2.1), and the subgroup $G_\lambda = \{g \in G | \mathrm{Ad}_g^* \lambda = \lambda\}$ freely acts on the regular level set $\mu^{-1}(\lambda)$.*

Consider the free loop space on M with time-dependent Lagrange multipliers,

$$\mathcal{L}_{\mathrm{MW}} := \{(\gamma, \xi) | \gamma \in C^\infty(S^1, M), \xi \in C^\infty(S^1, \mathfrak{g})\},$$

where S^1 is parametrized as $\mathbb{R}/\mathbb{Z}$. The loop space $\mathcal{L}_{\mathrm{MW}}$ is thought of as a Fréchet manifold with compact-open topology. We define here the action functional on the loop space $\mathcal{L}_{\mathrm{MW}}$ by

$$\mathcal{A}_{H,\mu,\lambda}(\gamma, \xi) := \int_0^1 \gamma^*\Theta - \int_0^1 (H(t, \gamma(t)) - (\mu(\gamma(t)) - \lambda, \xi(t)))\, \mathrm{d}t, \qquad (\gamma, \xi) \in \mathcal{L}_{\mathrm{MW}},$$

where $H_t \in C^\infty(M)$ is invariant under the G-action, and $\lambda \in \mathfrak{g}^*$ a regular value of μ. The tangent space to $\mathcal{L}_{\mathrm{MW}}$ at (γ, ξ) is canonically identified with $\Gamma(\gamma^* TM) \oplus C^\infty(S^1, \mathfrak{g})$. For $(X, \eta) \in T_{(\gamma,\xi)}\mathcal{L}_{\mathrm{MW}} = \Gamma(\gamma^* TM) \oplus C^\infty(S^1, \mathfrak{g})$, we take a smooth map $u : (-\varepsilon, \varepsilon) \times S^1 \to M; (s, t) \mapsto u(s, t)$ such that $u(0, t) = \gamma(t)$ and $(\partial_s u)(0, t) = X(t)$. The variation of the functional $\mathcal{A}_{H,\mu,\lambda}$ is then given by

$$(\mathrm{d}\mathcal{A}_{H,\mu,\lambda})_{(\gamma,\xi)}(X, \eta) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{A}_{H,\mu,\lambda}(u(s, \bullet), \xi + s\eta)$$
$$= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left\{ -\int_{[0,s]\times S^1} u^*\Omega - \int_0^1 (H(t, u(s, t))\right.$$

$$
\left. - (\mu(u(s,t)) - \lambda, \xi(t) + s\eta(t))) \, dt \right\}
$$

$$
= \int_0^1 \Omega_{\gamma(t)}(\dot{\gamma} - X_{H_t}(\gamma) + \xi_M^t(\gamma), X) \, dt + \int_0^1 (\mu(\gamma) - \lambda, \eta) \, dt, \qquad (3.1)
$$

where ξ_M^t is the fundamental vector field generated by $\xi(t) \in \mathfrak{g}$. Thus, $(\gamma, \xi) \in \mathcal{L}_{\mathrm{MW}}$ is a critical point of the action $\mathcal{A}_{H,\mu,\lambda}$ if and only if (γ, ξ) is subject to the equations

$$
\dot{\gamma} = X_{H_t}(\gamma) - \xi_M^t(\gamma), \qquad \mu(\gamma(t)) = \lambda. \qquad (3.2)
$$

This implies that if (γ, ξ) is a critical point of $\mathcal{A}_{H,\mu,\lambda}$, the orbit γ lies on the level set $\mu^{-1}(\lambda)$.

Lemma 3.2. *If $(\gamma, \xi) \in \mathcal{L}_{\mathrm{MW}}$ is a critical point of $\mathcal{A}_{H,\mu,\lambda}$, then $g(t)$ determined by $g^{-1}\dot{g} = \xi$ and $g(0) = \mathrm{id}$ is in G_λ. In particular, one has $\xi(t) \in \mathfrak{g}_\lambda := \mathrm{Lie}(G_\lambda)$ for $t \in S^1$.*

Proof. It is sufficient to prove that $g(t) \in G_\lambda$, namely, $\mathrm{Ad}_{g(t)}^*\lambda = \lambda$. By differentiating $\mu(g(t)\gamma(t))$ with respect to t and by using the G-equivariance of μ, we have

$$
\frac{d}{dt}\mu(g\gamma) = (d\mu)_{g\gamma}\big(g_*\dot{\gamma} + g_*\xi_M^t(\gamma)\big) = \mathrm{Ad}_g^*\left((d\mu)_\gamma(\dot{\gamma} + \xi_M^t(\gamma))\right) = \mathrm{Ad}_g^*(d\mu)_\gamma(X_{H_t}).
$$

According to this formula, we obtain, for any $\eta \in \mathfrak{g}$,

$$
\begin{aligned}
\frac{d}{dt}(\mu(g\gamma), \eta) &= (\mathrm{Ad}_g^*(d\mu)_\gamma(X_{H_t}), \eta) = ((d\mu)_\gamma(X_{H_t}), \mathrm{Ad}_{g^{-1}}\eta) \\
&= \Omega_{\gamma(t)}\big((\mathrm{Ad}_{g^{-1}}\eta)_M^t(\gamma), X_{H_t}(\gamma)\big) = -dH_t\big((\mathrm{Ad}_{g^{-1}}\eta)_M^t(\gamma)\big) = 0,
\end{aligned}
$$

where we have used the G-invariance of H_t. It then follows that $\mu(g\gamma)$ is preserved. Owing to $g(0) = \mathrm{id}$, the $g(t)$ is found to be in G_λ on account of

$$
\lambda = \mu(\gamma) = \mu(g\gamma) = \mathrm{Ad}_g^*\mu(\gamma) = \mathrm{Ad}_g^*\lambda.
$$

This completes the proof. □

Lemma 3.3. *If (γ, ξ) is a critical point of the action functional $\mathcal{A}_{H,\mu,\lambda}$, then γ projects to a one-periodic orbit $\pi_\lambda \circ \gamma$ of the reduced Hamiltonian system (2.5) on M_λ.*

Proof. Let (γ, ξ) be a critical point of $\mathcal{A}_{H,\mu,\lambda}$. Then, from (3.2), one has $\dot{\gamma} = X_{H_t}(\gamma) - \xi_M^t(\gamma)$ and $\mu(\gamma) = \lambda$, and further $\xi(t) \in \mathfrak{g}_\lambda$ from lemma 3.2. Differentiating a loop $\pi_\lambda \circ \gamma$ with respect to t, we have

$$
\frac{d}{dt}\pi_\lambda(\gamma(t)) = (\pi_\lambda)_*\big(X_{H_t}(\gamma(t)) - \xi_M^t(\gamma(t))\big) = X_{H_t^\lambda}(\pi_\lambda(\gamma(t))).
$$

This implies that the loop $\pi_\lambda \circ \gamma$ is a one-periodic orbit of the reduced Hamiltonian system (2.5) on $M_\lambda = \mu^{-1}(\lambda)/G_\lambda$. This ends the proof. □

We define an action of the infinite-dimensional group $\mathcal{G}_\lambda = C^\infty(S^1, G_\lambda)$ on $\mathcal{L}_{\mathrm{MW}}$ in the gauge-like manner

$$
g \cdot (\gamma, \xi) := (g\gamma, \mathrm{Ad}_g\xi - \dot{g}g^{-1}), \qquad g \in \mathcal{G}_\lambda, (\gamma, \xi) \in \mathcal{L}_{\mathrm{MW}}, \qquad (3.3)
$$

where $g \in \mathcal{G}_\lambda$ pointwise acts on $\gamma \in C^\infty(S^1, M)$; $(g\gamma)(t) = g(t)\gamma(t)$. Though the functional $\mathcal{A}_{H,\mu,\lambda}$ itself is not invariant under the action of $\mathcal{G}_\lambda$, it has the $\mathcal{G}_\lambda$-symmetry in the following sense.

Proposition 3.4

(1) Let $\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})$ denote the set of critical points of $\mathcal{A}_{H,\mu,\lambda}$. Then

$$
(\gamma, \xi) \in \mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda}) \Longrightarrow g \cdot (\gamma, \xi) \in \mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda}).
$$

(2) Let $(\gamma, \xi) \in \mathcal{L}_{\mathrm{MW}}$. The difference between the actions for (γ, ξ) and $g \cdot (\gamma, \xi)$ with $g \in \mathcal{G}_\lambda$ is given by

$$\mathcal{A}_{H,\mu,\lambda}(g \cdot (\gamma, \xi)) - \mathcal{A}_{H,\mu,\lambda}(\gamma, \xi) = \int_0^1 (\lambda, g^{-1}\dot{g})\, \mathrm{d}t. \tag{3.4}$$

(3) Let $(\gamma, \xi) \in \mathcal{L}_{\mathrm{MW}}$. Suppose that $\gamma : S^1 \to M$ lies on the level set $\mu^{-1}(\lambda)$. Then we have

$$\mathcal{A}_{H,\mu,\lambda}(g \cdot (\gamma, \xi)) = \mathcal{A}_{H,\mu,\lambda}(\gamma, \xi)$$

where g is in the identity component of $\mathcal{G}_\lambda$.

Proof.

(1) The proof runs straightforward as follows: if $\dot{\gamma} = X_{H_t}(\gamma) - \xi^t_M(\gamma)$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(g\gamma) &= g_*\dot{\gamma} + (\dot{g}g^{-1})^t_M(g\gamma) = g_*\big(X_{H_t}(\gamma) - \xi^t_M(\gamma)\big) + (\dot{g}g^{-1})^t_M(g\gamma)\\ &= X_{H_t}(g\gamma) - (\mathrm{Ad}_g\xi - \dot{g}g^{-1})^t_M(g\gamma).\end{aligned}$$

(2) Let $(\gamma, \xi) \in \mathcal{L}_{\mathrm{MW}}$ and $g \in \mathcal{G}_\lambda$. On account of $\mathrm{d}(g\gamma)/\mathrm{d}t = g_*\dot{\gamma} + g_*(g^{-1}\dot{g})^t_M(\gamma)$, we obtain

$$\begin{aligned}\mathcal{A}_{H,\mu,\lambda}(g \cdot (\gamma, \xi)) &= \int_0^1 \Theta_{g(t)\gamma(t)}(g_*\dot{\gamma} + g_*(g^{-1}\dot{g})^t_M(\gamma))\, \mathrm{d}t\\ &\quad - \int_0^1 (H(t, g(t)\gamma(t)) - (\mu(g(t)\gamma(t)) - \lambda, \mathrm{Ad}_g\xi - \dot{g}g^{-1}))\, \mathrm{d}t\\ &= \int_{S^1} \gamma^*\Theta - \int_0^1 (H(t, \gamma(t)) - (\mu(\gamma) - \lambda, \xi))\, \mathrm{d}t\\ &\quad + \int_0^1 \Theta_{\gamma(t)}\big((g^{-1}\dot{g})^t_M(\gamma)\big)\, \mathrm{d}t - \int_0^1 (\mu(\gamma(t)) - \lambda, g^{-1}\dot{g})\, \mathrm{d}t\\ &= \mathcal{A}_{H,\mu,\lambda}(\gamma, \xi) + \int_0^1 (\lambda, g^{-1}\dot{g})\, \mathrm{d}t,\end{aligned}$$

where we have used the G-invariance of Θ and H in the second equality, and definition (2.1) of the momentum map μ in the third equality.

(3) We show that if γ lies on $\mu^{-1}(\lambda)$ and g is contained in the identity component of $\mathcal{G}_\lambda$, then the right-hand side of (3.4) vanishes. From $\mu(\gamma) = \lambda$, it is put in the form

$$\begin{aligned}\int_0^1 (\lambda, g^{-1}\dot{g})\, \mathrm{d}t &= \int_0^1 (\mu(g\gamma), \dot{g}g^{-1})\, \mathrm{d}t = \int_0^1 \Theta_{g(t)\gamma(t)}((\dot{g}g^{-1})^t_M)\, \mathrm{d}t\\ &= \int_0^1 \Theta_{g(t)\gamma(t)}\left(\frac{\mathrm{d}}{\mathrm{d}t}(g\gamma) - g_*\dot{\gamma}\right)\mathrm{d}t\\ &= \int_{S^1} (g\gamma)^*\Theta - \int_{S^1} \gamma^*\Theta,\end{aligned}$$

where in the third equality, we have used the fact that $\mathrm{d}(g\gamma)/\mathrm{d}t = g_*\dot{\gamma} + (\dot{g}g^{-1})^t_M(g\gamma)$. Since g is in the identity component of $\mathcal{G}_\lambda$, there is a map $h : [0, 1] \times S^1 \to G_\lambda$ such that $h(0, t) = \mathrm{id}$ and $h(1, t) = g(t)$. With this h, we define $u : [0, 1] \times S^1 \to M$ by

$$u(s, t) = h(s, t)\gamma(t), \qquad (s, t) \in [0, 1] \times S^1, \tag{3.5}$$

which is subject to the boundary conditions $u(0, t) = \gamma(t)$ and $u(1, t) = g(t)\gamma(t)$. Then, from the Stokes theorem and $\Omega = -\mathrm{d}\Theta$, we obtain

$$\int_0^1 (\lambda, g^{-1}\dot{g})\, \mathrm{d}t = \int_{S^1} (g\gamma)^*\Theta - \int_{S^1} \gamma^*\Theta = -\int_{[0,1]\times S^1} u^*\Omega.$$

Since $u(s,t) \in \mu^{-1}(\lambda)$ if $\gamma(t) \in \mu^{-1}(\lambda)$, we have

$$\int_0^1 (\lambda, g^{-1}\dot{g})\,\mathrm{d}t = -\int_{[0,1]\times S^1} u^*\Omega = -\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\,(\pi_\lambda^*\Omega_\lambda)\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right)$$
$$= -\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\,\Omega_\lambda\left((\pi_\lambda)_*\frac{\partial u}{\partial s}, (\pi_\lambda)_*\frac{\partial u}{\partial t}\right) = 0.$$

In the last equality, we have used the fact that $(\pi_\lambda)_*\partial_s u$ vanishes, which is a consequence of definition (3.5) of u. Hence, the quantity $\int_0^1(\lambda, g^{-1}\dot{g})\,\mathrm{d}t$ vanishes. These prove the proposition. □

Now we stand in a position to state a main theorem.

Theorem 3.5 (Variational Method I). *Denote by* $\mathrm{Per}(H^\lambda)$ *the set of one-periodic orbits of the reduced system (2.5) on* M_λ. *Then there exists a one-to-one correspondence between* $\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})/\mathcal{G}_\lambda$ *and* $\mathrm{Per}(H_\lambda)$.

Proof. We have already obtained a projection

$$\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda}) \longrightarrow \mathrm{Per}(H^\lambda);\ (\gamma,\xi) \longmapsto \pi_\lambda \circ \gamma. \tag{3.6}$$

We show that the projection is surjective. Let ϕ^t be the flow of the original system (2.4) on M, and $\tilde{\gamma} : S^1 \to M_\lambda$ is a one-periodic orbit of the reduced system (2.5). Choose an arbitrary loop $\gamma : S^1 \to \mu^{-1}(\lambda)$ with $\pi_\lambda \circ \gamma = \tilde{\gamma}$. Then there exists $g(t)$ in G_λ such that $\phi^t(\gamma(0)) = g(t)\gamma(t)$ with $g(0) = \mathrm{id}$. Hence, we have

$$X_{H_t}(g(t)\gamma(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(g(t)\gamma(t)) = g(t)_*\dot{\gamma} + g(t)_*\xi^t_M(\gamma),$$

where $\xi = g^{-1}\dot{g}$. Since H_t is G-invariant, the above equation should be G-related to $\dot{\gamma} = X_{H_t}(\gamma) - \xi^t_M(\gamma)$ in order that $g(t)\gamma(t)$ is subject to the equation of motion on $\mu^{-1}(\lambda)$. Thus, our task is to prove that $\xi = g^{-1}\dot{g}$ is periodic, i.e., $\xi(t+1) = \xi(t)$ for all $t \in \mathbb{R}$. Since $\phi^t(\gamma(0)) = g(t)\gamma(t)$ is a relative periodic orbit of (2.4), there exists a $g_0 \in G_\lambda$ such that $g(t+1)\gamma(t+1) = g_0 g(t)\gamma(t)$ for all t. Since G_λ freely acts on $\mu^{-1}(\lambda)$, and since $\gamma(t+1) = \gamma(t)$, we have $g(t+1) = g_0 g(t)$, equivalently, $g(t+1) = g(1)g(t)$. This implies that $\xi = g^{-1}\dot{g}$ is periodic.

We now assume that $\pi_\lambda \circ \gamma_1 = \pi_\lambda \circ \gamma_2$ for $(\gamma_i, \xi_i) \in \mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})$, $i = 1, 2$. From the definition of π_λ, there is $g \in \mathcal{G}_\lambda$ such that $\gamma_2(t) = g(t)\gamma_1(t)$. Differentiation of this equation results in

$$X_{H_t}(g\gamma_1) - (\mathrm{Ad}_g\xi_1 - \dot{g}g^{-1})^t_M(g\gamma_1) = \frac{\mathrm{d}}{\mathrm{d}t}(g\gamma_1) = \dot{\gamma}_2 = X_{H_t}(\gamma_2) - (\xi_2)^t_M(\gamma_2).$$

Thus, it follows that $\xi_2 = \mathrm{Ad}_g\xi_1 - \dot{g}g^{-1}$, because G_λ freely acts on $\mu^{-1}(\lambda)$. Then (γ_1, ξ_1) and (γ_2, ξ_2) are $\mathcal{G}_\lambda$-equivalent. This implies that the map induced by (3.6),

$$\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})/\mathcal{G}_\lambda \longrightarrow \mathrm{Per}(H^\lambda);\ [\gamma,\xi] \longmapsto \pi_\lambda \circ \gamma, \tag{3.7}$$

is injective. It is clear that map (3.7) is surjective, since projection (3.6) is surjective. This ends the proof. □

Remark. We comment here on the relation between the present results and those by Ibort and Ontalba [IO96]. What they considered is the action functional (1.1) restricted to the space $C^\infty(S^1, \mu^{-1}(\lambda))$ to show that the restricted functional satisfies the similar proposition [IO96, proposition 5] as proposition 3.4. Their characterization for periodic orbits of the

Marsden–Weinstein reduced system (2.5) is to view such orbits as zeros of a one-form $\alpha_{H,\mu,\lambda}$ on $C^\infty(S^1, M_\lambda)$, which is defined to be

$$\Pi_\lambda^* \alpha_{H,\mu,\lambda} = d(\mathcal{A}_H|_{C^\infty(S^1;\mu^{-1}(\lambda))}),$$

where $\Pi_\lambda : C^\infty(S^1, \mu^{-1}(\lambda)) \to C^\infty(S^1, M_\lambda)$ is the projection induced by $\pi_\lambda : \mu^{-1}(\lambda) \to M_\lambda$, and d denotes the formal exterior derivation on $C^\infty(S^1, \mu^{-1}(\lambda))$.

We have to point out that functionals $A : C^\infty(S^1, \mu^{-1}(\lambda)) \to \mathbb{R}$ satisfying $\mathrm{d}A = \Pi_\lambda^* \alpha_{H,\mu,\lambda}$ are not unique. In fact, for $\xi \in C^\infty(S^1, \mathfrak{g}_\lambda)$, functionals of the form

$$A_\xi(\gamma) := \mathcal{A}_H(\gamma) + \int_0^1 \langle \mu(\gamma(t)), \xi(t) \rangle \, \mathrm{d}t, \qquad \gamma \in C^\infty(S^1; \mu^{-1}(\lambda))$$

satisfy $\mathrm{d}A_\xi = \Pi_\lambda^* \alpha_{H,\mu,\lambda}$, which means that critical points of A_ξ are projected to periodic orbits of (2.5) through Π_λ. Put another way, there are an infinite number of variational methods for periodic orbits of (2.5) which are parametrized by $\xi \in C^\infty(S^1, \mathfrak{g}_\lambda)$. In particular, Ibort and Ontalba's method is interpreted as that in the case $\xi \equiv 0$.

However, those methods should be unified because the reduced Hamiltonian system is unique. Our variational method (theorem 3.5) is a proposal for a unified characterization of periodic orbits of (2.5), where the parameter ξ of characterizations is taken as a Lagrange multiplier, and is considered as the freedom of gauge as well.

4. Variational method for the orbit reduced system in the case $\pi_1(\mathcal{O}) = 0$

In the previous section, we have established a variational method for finding periodic orbits of the Marsden–Weinstein reduced system (2.5). This section deals with a variational method for periodic orbits of the reduced system through the orbit reduction. A key to such a variational method is that there is a symplectic diffeomorphism

$$(M_\mathcal{O}, \Omega_\mathcal{O}) \cong (M \times \mathcal{O}) /\!\!/ G,$$

where $(M \times \mathcal{O}) /\!\!/ G$ denotes the symplectic quotient of the symplectic manifold $(M \times \mathcal{O}, \Omega \oplus (-\omega_\mathcal{O}))$ by the diagonal G-action $g \cdot (x, \lambda) = (gx, \mathrm{Ad}_g^* \lambda)$ with $g \in G$ and $(x, \lambda) \in M \times \mathcal{O}$. The momentum map with respect to the G-action on $M \times \mathcal{O}$ is given by

$$\mu \oplus (-\iota) : M \times \mathcal{O} \to \mathfrak{g}^*,$$

where $\iota : \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is the inclusion which is viewed as a momentum map with respect to the coadjoint action. This fact would imply that we could obtain the variational method for periodic orbits of the orbit reduced system (2.7), like theorem 3.5. In other words, we might apply the previous method in the case $\lambda = 0$ by replacing (M, Ω) and μ with $(M \times \mathcal{O}, \Omega \oplus (-\omega_\mathcal{O}))$ and $\mu \oplus (-\iota)$, respectively.

However, since the KKS form $\omega_\mathcal{O}$ is not always exact, the variational method in the previous section fails to work. Of course, it is a strong restriction to assume that $\omega_\mathcal{O}$ is an exact two-form. For example, if $G = \mathrm{SO}(3)$, then a generic coadjoint orbit $\mathcal{O}$ is two-sphere S^2, and $\omega_\mathcal{O}$, which proves to be an area form on $\mathcal{O} \cong S^2$, is not exact.

To take $\omega_\mathcal{O}$ into an action functional, we introduce the universal covering space of the free loop space on $\mathcal{O}$. For simplicity, we assume the following hypothesis.

Assumption 4.1

(1) A coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ passes through a regular value of μ, and G freely acts on $\mu^{-1}(\mathcal{O})$.

(2) $\mathcal{O}$ is simply connected, i.e., $\pi_1(\mathcal{O}) = 0$.

Condition (1) is needed to get the orbit reduction to work well and (2) is assumed in order to construct an action functional on the loop space of $M \times \mathcal{O}$ together with Lagrange multipliers.

Let $\mathcal{L}(\mathcal{O}) = C^\infty(S^1, \mathcal{O})$ be the free loop space on the coadjoint orbit $\mathcal{O}$, which consists of contractible loops on the assumption $\pi_1(\mathcal{O}) = 0$. The universal covering space of $\mathcal{L}(\mathcal{O})$ can be described as follows: for a fixed point $p_0 \in \mathcal{O}$, let $\widetilde{\mathcal{L}(\mathcal{O})}$ be

$$\widetilde{\mathcal{L}(\mathcal{O})} := \left\{ (\ell, \bar{\ell}) \,\middle|\, \begin{array}{l} \ell \in C^\infty(S^1, \mathcal{O}), \bar{\ell} \in C^\infty(D^2, \mathcal{O}) \\ \bar{\ell}(0) = p_0, \bar{\ell}(\mathrm{e}^{2\pi \mathrm{i} t}) = \ell(t), t \in \mathbb{R} \end{array} \right\} \Big/ \sim,$$

where $D^2 = \{z \in \mathbb{C} \,|\, |z| \leqslant 1\}$. The equivalence relation stated above is given by

$$(\ell_1, \bar{\ell}_1) \sim (\ell_2, \bar{\ell}_2) \quad \Longleftrightarrow \quad \begin{cases} \ell_1 = \ell_2, \\ \bar{\ell}_1 \#(-\bar{\ell}_2) \text{ is homotopic to a point,} \end{cases} \tag{4.1}$$

where $-\bar{\ell}_2$ is the disc with the orientation opposite to $\bar{\ell}_2$, and where $\bar{\ell}_1\#(-\bar{\ell}_2)$ denotes the glued sphere of two discs $\bar{\ell}_1, -\bar{\ell}_2$ along the boundary $\ell_1 = \ell_2$. We denote by $[\ell, \bar{\ell}]$ the equivalence class of a pair $(\ell, \bar{\ell})$. The space $\widetilde{\mathcal{L}(\mathcal{O})}$ is the universal covering space of the free loop space $\mathcal{L}(\mathcal{O})$, on which $\pi_2(\mathcal{O})$ acts in the manner, $A \cdot [\ell, \bar{\ell}] := [\ell, A\#\bar{\ell}]$ for $[\ell, \bar{\ell}] \in \widetilde{\mathcal{L}(\mathcal{O})}$ and $A \in \pi_2(\mathcal{O})$. Here the $A\#\bar{\ell} : D^2 \to \mathcal{O}$ is defined as follows: the element $A \in \pi_2(\mathcal{O})$ can be represented by a smooth map $A : S^2 = \mathbb{C} \cup \{\infty\} \to \mathcal{O}$ such that $A(0) = A(\infty) = p_0$. Then the smooth map $A\#\bar{\ell} : D^2 \to M$ is defined to be

$$(A\#\bar{\ell})(z) := \begin{cases} A\left(\dfrac{z}{1 - \rho_1(2|z|)}\right), & \text{if} \quad |z| \leqslant \dfrac{1}{2}, \\ \bar{\ell}\left(\rho_2(2|z| - 1)\dfrac{z}{|z|}\right), & \text{if} \quad \dfrac{1}{2} \leqslant |z| \leqslant 1, \end{cases}$$

where $\rho_1, \rho_2 : [0, 1] \to [0, 1]$ are non-decreasing smooth functions such that

$$\rho_1(r) = \begin{cases} r & \text{if} \quad 0 \leqslant r \leqslant 1 - 2\varepsilon, \\ 1 & \text{if} \quad 1 - \varepsilon \leqslant r \leqslant 1, \end{cases}$$

$$\rho_2(r) = \begin{cases} 0 & \text{if} \quad 0 \leqslant r \leqslant \varepsilon, \\ r & \text{if} \quad 2\varepsilon \leqslant r \leqslant 1, \end{cases}$$

for a sufficiently small $\varepsilon > 0$. Since $(A\#\bar{\ell})(0) = A(0) = p_0$ and $(A\#\bar{\ell})(\mathrm{e}^{2\pi \mathrm{i} t}) = \bar{\ell}(\mathrm{e}^{2\pi \mathrm{i} t}) = \ell(t)$, the pair $(\ell, A\#\bar{\ell})$ determines an element of $\widetilde{\mathcal{L}(\mathcal{O})}$. We do not write into $\widetilde{\mathcal{L}(\mathcal{O})}$ its dependence on $p_0 \in \mathcal{O}$, ρ_1 and ρ_2.

Now we define an action functional on the extended loop space,

$$\mathcal{L}_{\mathrm{O}} := \{(\gamma, [\ell, \bar{\ell}], \xi) \,|\, \gamma \in C^\infty(S^1, M), [\ell, \bar{\ell}] \in \widetilde{\mathcal{L}(\mathcal{O})}, \xi \in C^\infty(S^1, \mathfrak{g})\},$$

by

$$\mathcal{A}_{H,\mu,\mathcal{O}}(\gamma, [\ell, \bar{\ell}], \xi) := \int_{S^1} \gamma^*\Theta + \int_{D^2} \bar{\ell}^*\omega_{\mathcal{O}} - \int_0^1 (H(t, \gamma(t)) - (\mu(\gamma(t)) - \ell(t), \xi(t)))\,\mathrm{d}t.$$

From definition (4.1) of the equivalence relation, it is easy to see that this functional is independent of the choice of a representative of $[\ell, \bar{\ell}]$, and then well defined.

Lemma 4.2. *An element $(\gamma, [\ell, \bar{\ell}], \xi) \in \mathcal{L}_{\mathrm{O}}$ is a critical point of the action functional $\mathcal{A}_{H,\mu,\mathcal{O}}$ if and only if $(\gamma, [\ell, \bar{\ell}], \xi)$ is subject to the equations*

$$\dot{\gamma} = X_{H_t}(\gamma) - \xi^t_M(\gamma), \qquad \mu(\gamma(t)) = \ell(t), \qquad \dot{\ell} = -\mathrm{ad}^*_\xi \ell. \tag{4.2}$$

Thus, if $(\gamma, [\ell, \bar{\ell}], \xi)$ is a critical point of $\mathcal{A}_{H,\mu,\mathcal{O}}$, then γ lies on $\mu^{-1}(\mathcal{O})$.

Proof. We first remark that the tangent space $T_{(\gamma,[\ell,\bar{\ell}],\xi)}\mathcal{L}_{\mathrm{O}}$ is canonically identified with

$$\Gamma(\gamma^*TM) \oplus T_{[\ell,\bar{\ell}]}\widetilde{\mathcal{L}(\mathcal{O})} \oplus C^\infty(S^1,\mathfrak{g}) \cong \Gamma(\gamma^*TM) \oplus \Gamma(\ell^*T\mathcal{O}) \oplus C^\infty(S^1,\mathfrak{g}).$$

For $X \in \Gamma(\gamma^*TM)$, $Y \in \Gamma(\ell^*T\mathcal{O})$ and $\eta \in T_\xi C^\infty(S^1,\mathfrak{g}) = C^\infty(S^1,\mathfrak{g})$, we take up smooth maps $u : (-\epsilon,\epsilon) \times S^1 \to M$, $v : (-\epsilon,\epsilon) \times S^1 \to \mathcal{O}$ and $\bar{v} : (-\epsilon,\epsilon) \times D^2 \to \mathcal{O}$ such that

$$\begin{aligned} &u(0,t) = \gamma(t), && v(0,t) = \ell(t),\\ &\bar{v}(s,\mathrm{e}^{2\pi \mathrm{i}t}) = v(s,t), && \bar{v}(0,z) = \bar{\ell}(z),\\ &(\partial_s u)(0,t) = X(t), && (\partial_s v)(0,t) = Y(t), \end{aligned}$$

for $s \in (-\epsilon,\epsilon)$, $t \in S^1$ and $z \in D^2$. Note here that $\bar{v}(s,\bullet)$ is homotopic, with the boundary fixed, to the map

$$D^2 \longrightarrow \mathcal{O};\, z \longmapsto \begin{cases} \bar{v}(0,2z) = \bar{\ell}(0,2z) & \text{if} \quad |z| \leqslant 1/2,\\ v(s(2|z|-1),t) & \text{if} \quad 1/2 \leqslant |z| \leqslant 1 \text{ and } z = |z|\,\mathrm{e}^{2\pi \mathrm{i}t} \end{cases}$$

and thereby one has

$$\int_{D^2} \bar{v}(s,\bullet)^*\omega_{\mathcal{O}} = \int_{[0,s]\times S^1} v^*\omega_{\mathcal{O}} + \int_{D^2} \bar{\ell}^*\omega_{\mathcal{O}} \tag{4.3}$$

for any sufficiently small $|s| > 0$.

Since a smooth path

$$(-\epsilon,\epsilon) \longrightarrow \mathcal{L}_{\mathrm{O}};\, s \longmapsto (u(s,\bullet),[v(s,\bullet),\bar{v}(s,\bullet)],\xi + s\eta)$$

has the tangent vector (X,Y,η) at $s = 0$, the variation $(\mathrm{d}\mathcal{A}_{H,\mu,\mathcal{O}})_{(\gamma,[\ell,\bar{\ell}],\xi)}(X,Y,\eta)$ is calculated as

$$\begin{aligned} &(\mathrm{d}\mathcal{A}_{H,\mu,\mathcal{O}})_{(\gamma,[\ell,\bar{\ell}],\xi)}(X,Y,\eta)\\ &\quad = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{A}_{H,\mu,\mathcal{O}}(u(s,\bullet),[v(s,\bullet),\bar{v}(s,\bullet)],\xi + s\eta)\\ &\quad = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left\{-\int_{[0,s]\times S^1} u^*\Omega + \int_{[0,s]\times S^1} v^*\omega_{\mathcal{O}} + \int_{D^2} \bar{\ell}^*\omega_{\mathcal{O}}\right.\\ &\qquad \left. - \int_0^1 (H(t,u(s,t)) - (\mu(u(s,t)) - v(s,t),\xi(t) + s\eta(t)))\,\mathrm{d}t\right\}\\ &\quad = \int_0^1 \Omega_{\gamma(t)}\big(\dot{\gamma} - X_{H_t}(\gamma) + \xi_M^t(\gamma), X\big)\,\mathrm{d}t\\ &\qquad - \int_0^1 (\omega_{\mathcal{O}})_{\ell(t)}\big(\dot{\ell} + X_{h_t^\xi}(\ell), Y\big) + \int_0^1 (\mu(\gamma) - \ell,\eta)\,\mathrm{d}t, \end{aligned} \tag{4.4}$$

where h_t^ξ is a function on $\mathcal{O}$ given by $h_t^\xi(\lambda) = (\lambda,\xi(t))$, $\lambda \in \mathcal{O}$, and where $X_{h_t^\xi}$ is the Hamiltonian vector field associated with h_t^ξ on $\mathcal{O}$. In the second equality, we have used (4.3).

A straightforward computation shows that $X_{h_t^\xi}(\lambda) = \mathrm{ad}^*_{\xi(t)}(\lambda)$. Hence $(\gamma,[\ell,\bar{\ell},\xi])$ is a critical point of $\mathcal{A}_{H,\mu,\mathcal{O}}$ if and only if (4.2) holds. This ends the proof. □

These critical points of $\mathcal{A}_{H,\mu,\mathcal{O}}$ are related to one-periodic orbits of the reduced Hamilton equation (2.7) through the orbit reduction as follows.

Lemma 4.3. *If $(\gamma,[\ell,\bar{\ell}],\xi)$ is a critical point of the functional $\mathcal{A}_{H,\mu,\mathcal{O}}$, then γ projects to a one-periodic orbit $\pi_{\mathcal{O}} \circ \gamma$ of the reduced Hamilton equation (2.7).*

Proof. Assume that $(\gamma, [\ell, \bar{\ell}], \xi)$ is a critical point of the functional $\mathcal{A}_{H,\mu,\mathcal{O}}$. From lemma 4.2, it satisfies $\dot{\gamma} = X_{H_t}(\gamma) - \xi^t_M(\gamma)$. Thus, $\pi_{\mathcal{O}} \circ \gamma$ is subject to the reduced Hamilton equation $\mathrm{d}(\pi_{\mathcal{O}} \circ \gamma)/\mathrm{d}t = X_{H^{\mathcal{O}}_t}(\pi_{\mathcal{O}} \circ \gamma)$, because of (2.8) and $(\pi_{\mathcal{O}})_*(\xi^t_M) = 0$. This completes the proof. □

The loop space $\mathcal{L}_{\mathrm{O}}$ admits a natural action of the group $\pi_2(\mathcal{O})$ in the manner

$$A \cdot (\gamma, [\ell, \bar{\ell}], \xi) := (\gamma, A \cdot [\ell, \bar{\ell}], \xi) = (\gamma, [\ell, A\#\bar{\ell}], \xi) \tag{4.5}$$

for $A \in \pi_2(\mathcal{O})$. The quotient space $\mathcal{L}_{\mathrm{O}}/\pi_2(\mathcal{O})$ is clearly identified with

$$\mathcal{L}_{\mathrm{O}}/\pi_2(\mathcal{O}) = \left\{(\gamma, \ell, \xi) | \gamma \in C^\infty(S^1, M), \ell \in C^\infty(S^1, \mathcal{O}), \xi \in C^\infty(S^1, \mathfrak{g})\right\},$$

on which the infinite-dimensional group $\mathcal{G} = C^\infty(S^1, G)$ acts in the manner

$$g \cdot (\gamma, \ell, \xi) := (g\gamma, \mathrm{Ad}^*_g \ell, \mathrm{Ad}_g \xi - \dot{g}g^{-1}), \qquad g \in \mathcal{G}, \tag{4.6}$$

where the pointwise action of $g \in \mathcal{G}$ on ℓ is defined by $(\mathrm{Ad}^*_g \ell)(t) = \mathrm{Ad}_{g(t)} \ell(t)$.

The action functional $\mathcal{A}_{H,\mu,\mathcal{O}}$ has the $\pi_2(\mathcal{O})$ and $\mathcal{G}$-symmetry in the following sense.

Proposition 4.4

(1) Let $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})$ *denote the set of critical points of* $\mathcal{A}_{H,\mu,\mathcal{O}}$. *Then, action (4.5) of* $\pi_2(\mathcal{O})$ *leaves* $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})$ *invariant, that is,*

$$\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}}) \ni (\gamma, [\ell, \bar{\ell}], \xi) \longmapsto A \cdot (\gamma, [\ell, \bar{\ell}], \xi) \in \mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}}),$$

for any $A \in \pi_2(\mathcal{O})$. *This implies that there is a natural projection of* $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})$ *to*

$$\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}) \cong \left\{(\gamma, \ell, \xi) \left| \begin{array}{l} \gamma \in C^\infty(S^1, M), \ell \in C^\infty(S^1, \mathcal{O}), \xi \in C^\infty(S^1, \mathfrak{g}^*), \\ \dot{\gamma} = X_{H_t}(\gamma) - \xi^t_M(\gamma), \mu(\gamma(t)) = \ell(t), \dot{\ell} = -\mathrm{ad}_\xi \ell, \end{array} \right.\right\}.$$

Furthermore, $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})$ *is invariant under action (4.6) of* $\mathcal{G}$.

(2) Let $(\gamma, [\ell, \bar{\ell}], \xi) \in \mathcal{L}_{\mathrm{O}}$. *Then we have, for any* $A \in \pi_2(\mathcal{O})$,

$$\mathcal{A}_{H,\mu,\mathcal{O}}(A \cdot (\gamma, [\ell, \bar{\ell}], \xi)) = \mathcal{A}_{H,\mu,\mathcal{O}}(\gamma, [\ell, \bar{\ell}], \xi) + \omega_{\mathcal{O}}(A),$$

where $\omega_{\mathcal{O}}(A)$ *denotes the integral of* $\omega_{\mathcal{O}}$ *over the sphere* $A : S^2 \to \mathcal{O}$.

Proof.

(1) From lemma 4.2, it is obvious that $A \cdot (\gamma, [\ell, \bar{\ell}], \xi)$ is a critical point of $\mathcal{A}_{H,\mu,\mathcal{O}}$ if $(\gamma, [\ell, \bar{\ell}], \xi)$ is so.

If (γ, ℓ, ξ) is in $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})$, it satisfies (4.2). We can easily check that $g \cdot (\gamma, \ell, \xi)$ also satisfies (4.2), so that $g \cdot (\gamma, \ell, \xi)$ is in $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})$.

(2) To prove the assertion, it is sufficient to show that $\int_{D^2} (A\#\bar{\ell})^* \omega_{\mathcal{O}} = \int_{D^2} \bar{\ell}^* \omega_{\mathcal{O}} + \omega_{\mathcal{O}}(A)$. From the definition of $A\#\bar{\ell}$, we obtain

$$\begin{aligned} \int_{|z|\geqslant 1/2} (A\#\bar{\ell})^* \omega_{\mathcal{O}} &= \int_{1/2}^1 \mathrm{d}r' \int_0^1 \mathrm{d}t\, \omega_{\mathcal{O}} \left(\frac{\partial}{\partial r'} \ell(\rho_2(2r'-1)\,\mathrm{e}^{2\pi i t}), \frac{\partial}{\partial t} \ell(\rho_2(2r-1)\,\mathrm{e}^{2\pi i t}) \right) \\ &= \int_0^1 \mathrm{d}r \int_0^1 \mathrm{d}t\, \omega_{\mathcal{O}} \left(\frac{\partial \ell}{\partial r}(r\,\mathrm{e}^{2\pi i t}), \frac{\partial \ell}{\partial t}(r\,\mathrm{e}^{2\pi i t}) \right) \\ &= \int_{D^2} \bar{\ell}^* \omega_{\mathcal{O}}. \end{aligned}$$

Further, the restriction of $A\#\bar{\ell}$ to $\{z \in \mathbb{C} | |z| \leqslant 1/2\}$ is regarded as a map from S^2 to $\mathcal{O}$, which is homotopic to A. Thus, the integral $\int_{|z|\leqslant 1/2} (A\#\bar{\ell})^* \omega_{\mathcal{O}}$ is equal to $\omega_{\mathcal{O}}(A)$. Put together, these equations show that $\int_{D^2} (A\#\bar{\ell})^* \omega_{\mathcal{O}} = \int_{D^2} \bar{\ell}^* \omega_{\mathcal{O}} + \omega_{\mathcal{O}}(A)$. □

The action functional $\mathcal{A}_{H,\mu,\mathcal{O}}$ has another symmetry, which is corresponding with proposition 3.4 (3). To prove this, we need some preparation.

Let $(\gamma, [\ell, \bar{\ell}], \xi) \in \mathcal{L}_{\mathrm{O}}$, and $g \in \mathcal{G}$ contained in the identity component of the group $\mathcal{G}$. Then, there is a map $h : [0, 1] \times S^1 \to G$ such that $h(0, t) = \mathrm{id}$ and $h(1, t) = g(t)$. With this h, we can choose a map $\bar{\ell}' : D^2 \to \mathcal{O}$ which satisfies $\bar{\ell}'(\mathrm{e}^{2\pi \mathrm{i}t}) = \mathrm{Ad}^*_{g(t)}\ell(t)$, and is homotopic to the map

$$D^2 \longrightarrow \mathcal{O}; z \longmapsto \begin{cases} \bar{\ell}(2z) & \text{if} \quad 0 \leqslant |z| \leqslant 1/2, \\ \mathrm{Ad}^*_{h(2|z|-1,t)}\ell(t) & \text{if} \quad 1/2 \leqslant |z| \leqslant 1 \text{ and } z = |z|\,\mathrm{e}^{2\pi \mathrm{i}t}, \end{cases} \tag{4.7}$$

with the boundary fixed. Note here that since $\bar{\ell}'$ is homotopic to map (4.7), the integral $\int_{D^2}(\bar{\ell}')^*\omega_{\mathcal{O}}$ is equal to

$$\int_{D^2}(\bar{\ell}')^*\omega_{\mathcal{O}} = \int_{D^2}\bar{\ell}^*\omega_{\mathcal{O}} + \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\, \omega_{\mathcal{O}}\left(\frac{\partial}{\partial s}\mathrm{Ad}^*_{h(s,t)}\ell(t), \frac{\partial}{\partial t}\mathrm{Ad}^*_{h(s,t)}\ell(t)\right). \tag{4.8}$$

Now we are in a position to state the $\mathcal{G}$-symmetry of the action functional $\mathcal{A}_{H,\mu,\mathcal{O}}$ as follows.

Proposition 4.5. *Let $(\gamma, [\ell, \bar{\ell}], \xi) \in \mathcal{L}_{\mathrm{O}}$ satisfying $\mu(\gamma(t)) = \ell(t)$, and $g \in \mathcal{G}$ be in the identity component of $\mathcal{G}$. Choose a smooth map $\bar{\ell}' : D^2 \to \mathcal{O}$ which satisfies $\bar{\ell}'(\mathrm{e}^{2\pi \mathrm{i}t}) = \mathrm{Ad}^*_{g(t)}\ell(t)$, and which is homotopic, with the boundary fixed, to the map defined by (4.7). Then the equality*

$$\mathcal{A}_{H,\mu,\mathcal{O}}(g\gamma, [\mathrm{Ad}_g\ell, \bar{\ell}'], \mathrm{Ad}_g\xi - \dot{g}g^{-1}) = \mathcal{A}_{H,\mu,\mathcal{O}}(\gamma, [\ell, \bar{\ell}], \xi) \tag{4.9}$$

holds.

Proof. In the same manner as in the proof of proposition 3.4 (3), the difference between the quantities on the left- and right-hand side of (4.9) proves to be expressed as

$$\begin{aligned} &\mathcal{A}_{H,\mu,\mathcal{O}}(g\gamma, [\mathrm{Ad}_g\ell, \bar{\ell}'], \mathrm{Ad}_g\xi - \dot{g}g^{-1}) - \mathcal{A}_{H,\mu,\mathcal{O}}(\gamma, [\ell, \bar{\ell}], \xi) \\ &\qquad = \int_0^1 \Theta_{\gamma(t)}((g^{-1}\dot{g})^t_M)\,\mathrm{d}t + \int_{D^2}(\bar{\ell}')^*\omega_{\mathcal{O}} - \int_{D^2}\bar{\ell}^*\omega_{\mathcal{O}}. \end{aligned} \tag{4.10}$$

Here we define $u : [0, 1] \times S^1 \to M$ to be

$$u(s, t) = h(s, t)\gamma(t), \qquad (s, t) \in [0, 1] \times S^1.$$

In the same discussion as in the proof of proposition 3.4 (3), the quantity $\int_0^1 \Theta_{\gamma(t)}((g^{-1}\dot{g})^t_M)\,\mathrm{d}t$ is brought into the form

$$\begin{aligned} \int_0^1 \Theta_{\gamma(t)}((g^{-1}\dot{g})^t_M)\,\mathrm{d}t &= -\int_{[0,1]\times S^1} u^*\Omega = -\int_{[0,1]\times S^1} u^*(\pi^*_{\mathcal{O}}\Omega_{\mathcal{O}} + \mu^*\omega_{\mathcal{O}}) \\ &= -\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\, \Omega_{\mathcal{O}}\left((\pi_{\mathcal{O}})_*\frac{\partial u}{\partial s}, (\pi_{\mathcal{O}})_*\frac{\partial u}{\partial t}\right) \\ &\quad - \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\, \omega_{\mathcal{O}}\left(\frac{\partial}{\partial s}\mu \circ u, \frac{\partial}{\partial t}\mu \circ u\right), \end{aligned}$$

where we have used the fact that $u(s, t) = h(s, t)\gamma(t) \in \mu^{-1}(\mathcal{O})$ and definition (2.6) of $\Omega_{\mathcal{O}}$ in the second equality. The definition of u implies that $(\pi_{\mathcal{O}})_*\partial_s u$ vanishes. Thus, using the assumption $\mu(\gamma(t)) = \ell(t)$ together with (4.8), we can get

$$\begin{aligned} \int_0^1 \Theta_{\gamma(t)}((g^{-1}\dot{g})^t_M)\,\mathrm{d}t &= -\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\, \omega_{\mathcal{O}}\left(\frac{\partial}{\partial s}\mathrm{Ad}^*_{h(s,t)}\ell(t), \frac{\partial}{\partial t}\mathrm{Ad}^*_{h(s,t)}\ell(t)\right) \\ &= -\int_{D^2}(\bar{\ell}')^*\omega_{\mathcal{O}} + \int_{D^2}\bar{\ell}^*\omega_{\mathcal{O}}. \end{aligned} \tag{4.11}$$

Hence, (4.10) and (4.11) are put together to result in (4.9). This ends the proof. □

Like theorem 3.5, $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})$ projects to the set of one-periodic orbits of the reduced system (2.7).

Theorem 4.6 (Variational Method II)**.** *Denote by* $\mathrm{Per}(H^{\mathcal{O}})$ *the set of one-periodic orbits of the reduced system (2.7) through the orbit reduction. Then, under assumption 4.1, there exists a one-to-one correspondence*

$$\frac{\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})}{\mathcal{G}} \quad\cong\quad \mathrm{Per}(H^{\mathcal{O}}).$$

Proof. The proof of the theorem runs in parallel to that of theorem 3.5. First of all, lemma 4.3 shows that there exists the projection from $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})$ to $\mathrm{Per}(H^{\mathcal{O}})$,

$$\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}) \longrightarrow \mathrm{Per}(H^{\mathcal{O}});\ (\gamma,\ell,\xi) \longmapsto \pi_{\mathcal{O}}\circ\gamma. \tag{4.12}$$

We show that this map (4.12) is surjective. Let $\tilde{\gamma}$ be a one-periodic orbit of the reduce system (2.7), and $\gamma : S^1 \to \mu^{-1}(\mathcal{O})$ be any lift of $\tilde{\gamma}$. For the flow ϕ^t of the original system (2.4), we can find $g(t) \in G$ such that $\phi^t(\gamma(0)) = g(t)\gamma(t)$. By the same discussion as in theorem 3.5, $\xi := g^{-1}\dot{g}$ is proved to be periodic in t, and the equation $\dot{\gamma} = X_{H_t}(\gamma) - \xi^t_M(\gamma)$ holds true. Furthermore, the $\ell(t) := \mu(\gamma(t))$ is subject to

$$\dot{\ell} = -\mathrm{ad}^*_\xi \ell.$$

In fact, we can obtain this equation by differentiating $\mathrm{Ad}^*_{g(t)}\ell(t) = \mu(g(t)\gamma(t)) = \mu(\phi^t(\gamma(0))) = \mu(\gamma(0))$. Because of assumption 4.1, we can find $\bar{\ell} : D^2 \to \mathcal{O}$ which bounds ℓ as its boundary. Hence, the triple $(\gamma, [\ell,\bar{\ell}], \xi)$ constructed above is a critical point of $\mathcal{A}_{H,\mu,\mathcal{O}}$, and projects through $\pi_{\mathcal{O}}$ to $\tilde{\gamma} = \pi_{\mathcal{O}}\circ\gamma$. This means the surjectivity of (4.12).

Next, we prove that $(\gamma_i, \ell_i, \xi_i) \in \mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}), i = 1, 2$, are $\mathcal{G}$-equivariant if $\pi_{\mathcal{O}}\circ\gamma_1 = \pi_{\mathcal{O}}\circ\gamma_2$. From the definition of $\pi_{\mathcal{O}}$, there is $g(t) \in G$ satisfying $\gamma_2(t) = g(t)\gamma_1(t)$, so that we have

$$X_{H_t}(\gamma_2) - \xi_2(\gamma_2) = \dot{\gamma}_2 = \frac{\mathrm{d}}{\mathrm{d}t}(g\gamma_1) = X_{H_t}(g\gamma_1) - (\mathrm{Ad}_g\xi - \dot{g}g^{-1})^t_M(g\gamma_1).$$

The assumption of the free action of G on $\mu^{-1}(\mathcal{O})$ implies that $\xi_2 = \mathrm{Ad}^*_g\xi - \dot{g}g^{-1}$. From $\ell_i = \mu(\gamma_i)$ together with $\gamma_2 = g\gamma_1$, it follows that $\ell_2 = \mathrm{Ad}^*_g\ell_1$. Thus, $(\gamma_1, \ell_1, \xi_1)$ is related to $(\gamma_2, \ell_2, \xi_2)$ by $(\gamma_2, \ell_2, \xi_2) = g\cdot(\gamma_1, \ell_1, \xi_1)$. This implies that projection (4.12) yields to a well-defined injective map

$$\frac{\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})}{\mathcal{G}} \longrightarrow \mathrm{Per}(H^{\mathcal{O}});\ (\gamma,[\ell,\bar{\ell}],\xi) \longmapsto \pi_{\mathcal{O}}\circ\gamma.$$

This map is also surjective because the map (4.12) is so. This completes the proof. □

Theorem 4.7. *Let* $\mathcal{O}$ *be a coadjoint orbit in* $\mathfrak{g}^*$ *through a regular value* $\lambda \in \mathfrak{g}^*$ *of the momentum map* $\mu : M \to \mathfrak{g}^*$*. Then* $(\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}))/\mathcal{G}$ *can be identified with* $\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})/\mathcal{G}_\lambda$ *under hypothesis 4.1.*

Proof. We directly show the assertion by constructing bijections between $\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})/\mathcal{G}_\lambda$ and $(\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}))/\mathcal{G}$.

A critical point (γ, ξ) of $\mathcal{A}_{H,\mu,\lambda}$ is naturally regarded as a critical point of $\mathcal{A}_{H,\mu,\mathcal{O}}$. Indeed, by (3.2) and lemma 4.2, $(\gamma, [\ell_\lambda, \bar{\ell}_\lambda], \xi)$ is a critical point of $\mathcal{A}_{H,\mu,\mathcal{O}}$, where ℓ_λ and $\bar{\ell}_\lambda$ are the constant maps from S^1 and D^2 to λ, respectively. Thus, this correspondence induces a well-defined map

$$i : \frac{\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})}{\mathcal{G}_\lambda} \longrightarrow \frac{\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})}{\mathcal{G}};\ [\gamma,\xi] \longmapsto [\gamma,\ell_\lambda,\xi],$$

where $[\gamma, \ell_\lambda, \xi]$ is the equivalence class with the representative $(\gamma, \ell_\lambda, \xi)$. Note here that the map i is well defined. In fact, for $(\gamma, \xi) \in \mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})$ and $g \in \mathcal{G}_\lambda \subset \mathcal{G}$, one has

$$i([g \cdot (\gamma, \xi)]) = [g\gamma, \ell_\lambda, \mathrm{Ad}_g\xi - \dot{g}g^{-1}] = [g \cdot (\gamma, \ell_\lambda, \xi)] = i([\gamma, \xi]).$$

Next, we construct a map from $(\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}))/\mathcal{G}$ to $\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})/\mathcal{G}_\lambda$. Let (γ, ℓ, ξ) be in $\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})$. Without loss of generality, we may suppose that $\mu(\gamma(0)) = \ell(0) = \lambda$. Otherwise, we can replace (γ, ℓ, ξ) by $(k\gamma, \mathrm{Ad}^*_k\ell, \mathrm{Ad}_k\xi)$, where k is an element of G such that $\mu(\gamma(0)) = \ell(0) = \mathrm{Ad}^*_{k^{-1}}\lambda$, and is viewed as a constant map $S^1 \to G; t \mapsto k$.

Now we define a path $t \mapsto g(t)$ in G as the solution of the differential equation $g^{-1}\dot{g} = \xi$ with the initial condition $g(0) = \mathrm{id}$. Since ξ is one-periodic in t, we have $g(t+1) = g(1)g(t)$. The $g\gamma$ is subject to the Hamilton equation (2.4)

$$\frac{\mathrm{d}}{\mathrm{d}t} g\gamma = X_{H_t}(g\gamma) - (\mathrm{Ad}_g\xi - \dot{g}g^{-1})^t_M(g\gamma) = X_{H_t}(g\gamma),$$

and then lies on the level set $\mu^{-1}(\gamma(0)) = \mu^{-1}(\lambda)$. However, the pair $(g\gamma, 0)$ is not a critical point of $\mathcal{A}_{H,\mu,\lambda}$ because $g\gamma$ is not a loop in $\mu^{-1}(\lambda)$. In what follows, we construct $h(t) \in G_\lambda$ such that $h^{-1}g\gamma$ becomes a loop in $\mu^{-1}(\lambda)$.

Let $\eta_0 \in \mathfrak{g}$ be an arbitrary element independent of t, and $\tau : [0, 1] \to [0, 1]$ be a smooth map satisfying

$$\tau(t) = \begin{cases} 0 & \text{if} \quad 0 \leqslant t \leqslant \epsilon, \\ 1 & \text{if} \quad 1 - \epsilon \leqslant t \leqslant 1, \end{cases}$$

where $\epsilon > 0$ is a sufficiently small constant. Using $\tau(t)$, we put

$$\tilde{\xi}(t) = \eta_0 + \dot{\tau}(t)\,\mathrm{e}^{-(t-\tau(t))\eta_0}(\xi(\tau(t)) - \eta_0)\,\mathrm{e}^{(t-\tau(t))\eta_0}, \qquad t \in [0, 1].$$

From $\xi(t+1) = \xi(t)$ and from the definition of $\tau(t)$, $\tilde{\xi}(t)$ is equal to η_0 for $0 \leqslant t \leqslant \epsilon$ and $1 - \epsilon \leqslant t \leqslant 1$, so that $\tilde{\xi}$ is a periodic map from $\mathbb{R}$ to $\mathfrak{g}$ of period one, which we may view as $\tilde{\xi} : S^1 \to \mathfrak{g}$.

Lemma 4.8. *Define a map $h : \mathbb{R} \to G$ to be a solution of $h^{-1}\dot{h} = \tilde{\xi}$ under the initial condition $h(0) = \mathrm{id}$. The map h is explicitly written as*

$$h(t) = g([t] + \tau(t - [t]))\,\mathrm{e}^{(t-[t]-\tau(t-[t]))\eta_0},$$

where $[t]$ denotes the integer which is maximal among integers not greater than t. In particular, we have $h(t+1) = g(1)h(t)$.

We postpone the proof of this lemma, and proceed with the proof of the theorem.

Since $g(t+1) = g(1)g(t)$ and $h(t+1) = g(1)h(t)$, $\gamma' := h^{-1}g\gamma$ and $\xi' := \mathrm{Ad}_{h^{-1}g}\xi - \mathrm{d}(h^{-1}g)/\mathrm{d}t \cdot (h^{-1}g)^{-1}$ are found to be periodic in t. Further, it can be easily verified that (γ', ξ') is a critical point of $\mathcal{A}_{H,\mu,\lambda}$. Hence we can define a map

$$\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}) \longrightarrow \mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda}); (\gamma, \ell, \xi) \longmapsto (\gamma', \xi'). \qquad (4.13)$$

We show that map (4.13) projects a map from $(\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}))/\mathcal{G}$ to $\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})/\mathcal{G}_\lambda$. Let $(\gamma, \ell, \xi) \in \mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})$ with $\mu(\gamma(0)) = \lambda$ and $k \in \mathcal{G}$. Without loss of generality, we may assume that $k(0) = \mathrm{id}$. As was done in the above, we bring $k \cdot (\gamma, \ell\xi) = (k\gamma, \mathrm{Ad}^*_k\ell, \mathrm{Ad}_k\xi - k^{-1}\dot{k})$ into a relative periodic orbit $\tilde{g}k\gamma$ lying on $\mu^{-1}(\lambda)$, where $\tilde{g}$ is the solution of $\tilde{g}^{-1}\dot{\tilde{g}} = \mathrm{Ad}_k\xi - k^{-1}\dot{k}$ with $\tilde{g}(0) = \mathrm{id}$. Since both $g\gamma$ and $\tilde{g}k\gamma$ are subject to the Hamilton equation (2.4) and since $g(0)\gamma(0) = \tilde{g}(0)k(0)\gamma(0) = \gamma(0)$, they should coincide with each other. This implies that (4.13) maps (γ, ℓ, ξ) and $k \cdot (\gamma, \ell, \xi)$ into the same critical point of $\mathcal{A}_{H,\mu,\lambda}$. Thus, map (4.13) canonically induces the map

$$j : \frac{\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O})}{\mathcal{G}} \longrightarrow \frac{\mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})}{\mathcal{G}_\lambda}; [\gamma, \ell, \xi] \longmapsto [\gamma', \xi'].$$

Finally, we prove that i and j are bijections. The equality $j \circ i = \mathrm{id}$ is easy to prove from

$$(j \circ i)([\gamma, \xi]) = j([\gamma, \ell_\lambda, \xi]) = [\gamma, \xi], \qquad [\gamma, \xi] \in \mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda}).$$

In order to show that $i \circ j = \mathrm{id}$, we continue to use the same notations g, h and (γ', ξ') as in (4.13). Since $h^{-1}g$ satisfies $\mathrm{Ad}^*_{h(t)^{-1}g(t)}\ell(t) = \lambda$ because of the construction of h, one obtains

$$(i \circ j)([\gamma, \ell, \xi]) = i([\gamma', \xi']) = [\gamma', \ell_\lambda, \xi'] = [\gamma, \mathrm{Ad}^*_{g^{-1}h}\ell_\lambda, \xi] = [\gamma, \ell, \xi],$$

which proves that $i \circ j = \mathrm{id}$. This proves that i and j are bijectives.

To complete the proof of theorem 4.7, we need to prove lemma 4.8.

Proof of lemma 4.8. A straightforward calculation provides a solution $h(t)$ for $t \in [0, 1]$ in the form

$$h(t) = g(\tau(t))\,\mathrm{e}^{(t-\tau(t))\eta_0} \qquad \text{for} \quad t \in [0, 1].$$

In particular, one has $h(1) = g(1)$. Since $\tilde{\xi}$ is one-periodic in t, we have $h(t+1) = h(1)h(t) = g(1)h(t)$. Thus, for any $t \in \mathbb{R}$, $h(t)$ is expressed as

$$\begin{aligned} h(t) &= g([t])h(t-[t]) = g([t])g(\tau(t-[t]))\,\mathrm{e}^{(t-[t]-\tau(t-[t]))\eta_0} \\ &= g([t] + \tau(t-[t]))\,\mathrm{e}^{(t-[t]-\tau(t-[t]))\eta_0}. \end{aligned}$$

This proves lemma 4.8, and completes the proof of theorem 4.7.

Remark. The map $j : (\mathrm{Crit}(\mathcal{A}_{H,\mu,\mathcal{O}})/\pi_2(\mathcal{O}))/\mathcal{G} \to \mathrm{Crit}(\mathcal{A}_{H,\mu,\lambda})/\mathcal{G}_\lambda$ defined in the proof of the theorem is independent of the choice of $\tau : [0, 1] \to [0, 1]$ and $\eta_0 \in \mathfrak{g}_\lambda$, while the map (4.13) depends on them.

5. Concluding remarks

We comment on the variational method for relative periodic orbits of the Hamiltonian system (2.4) in section 2. In particular, we explain why the boundary condition $\Theta_{\gamma(1)}(X(1)) - \Theta_{\gamma(0)}(X(0)) = 0$ is necessary, in comparison with the usual variational method for paths in the case where symplectic manifolds are cotangent bundles T^*P.

Let us make a brief review of the variational method on the cotangent bundle T^*P. See also [Ar89]. Let θ be the Liouville one-form of T^*P, and $p_0, p_1 \in P$ be fixed points. On the path space

$$\mathcal{P}(p_0, p_1) = \{\ell : [0, 1] \to T^*P \,|\, \ell(0) \in T^*_{p_0}P, \ell(1) \in T^*_{p_1}P\},$$

the action functional is defined as

$$\mathcal{A}(\ell) = \int_{[0,1]} \ell^*\theta - \int_0^1 H(t, \ell(t))\,\mathrm{d}t.$$

For $\ell \in \mathcal{P}(p_0, p_1)$, we take a smooth map $u : (-\epsilon, \epsilon) \times [0, 1] \to T^*P$ such that

$$u(0, t) = \ell(t), \qquad u(s, 0) \in T^*_{p_0}P \text{ and } u(s, 1) \in T^*_{p_1}P,$$

for any $s \in (-\epsilon, \epsilon)$ and $t \in [0, 1]$. Then the variational vector field $X \in \Gamma(\ell^*T(T^*P))$ along ℓ associated with u is determined by $X = (\partial_s u)|_{s=0}$. The first variational formula of $\mathcal{A}$ is then expressed as

$$
\begin{aligned}
(\mathrm{d}\mathcal{A})_\ell(X) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{A}(u(s,\bullet)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left\{ \int_0^1 \theta_{u(s,t)}\left(\frac{\partial u}{\partial t}\right)\mathrm{d}t - \int_0^1 \theta_{u(0,t)}\left(\frac{\partial u}{\partial t}\right)\mathrm{d}t \right. \\
&\quad - \int_0^s \theta_{u(s',1)}\left(\frac{\partial u}{\partial s'}\right)\mathrm{d}s' + \int_0^s \theta_{u(s',0)}\left(\frac{\partial u}{\partial s'}\right)\mathrm{d}s' \\
&\quad \left. + \int_0^s \theta_{u(s',1)}\left(\frac{\partial u}{\partial s'}\right)\mathrm{d}s' - \int_0^s \theta_{u(s',0)}\left(\frac{\partial u}{\partial s'}\right)\mathrm{d}s' - \int_0^1 H(t,u(s,t))\,\mathrm{d}t \right\} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left\{ -\int_{[0,s]\times[0,1]} u^*\omega - \int_0^1 H(t,u(s,t))\,\mathrm{d}t \right. \\
&\quad \left. + \int_0^s \theta_{u(s',1)}\left(\frac{\partial u}{\partial s'}\right)\mathrm{d}s' - \int_0^s \theta_{u(s',0)}\left(\frac{\partial u}{\partial s'}\right)\mathrm{d}s' \right\} \\
&= \int_0^1 \omega_{\ell(t)}(\dot\ell - X_{H_t}, X)\,\mathrm{d}t + \theta_{\ell(1)}(X(1)) - \theta_{\ell(0)}(X(0)).
\end{aligned}
$$

Here we have used the Stokes theorem and $\omega = -\mathrm{d}\theta$ in the third equality. The boundary condition $u(s,t) \in T^*_{p_t}P, t = 0, 1$, implies that $X(t) \in T_{\ell(t)}(T^*P), t = 0, 1$, projects to zero through the natural projection $p_* : T(T^*P) \to TP$. Thus, from the definition of the Liouville one-form θ, the quantities $\theta_{\ell(0)}(X(0))$ and $\theta_{\ell(1)}(X(1))$ vanish. Hence we obtain the first variational formula

$$
(\mathrm{d}\mathcal{A})_\ell(X) = \int_0^1 \omega_{\ell(t)}(\dot\ell - X_{H_t}, X)\,\mathrm{d}t,
$$

which implies that critical points of $\mathcal{A}$ are subject to the Hamilton equation (2.4).

This calculation illustrates that though the Neumann condition $\theta_{\ell(0)}(X(0)) - \theta_{\ell(1)}(X(1)) = 0$ for variational vectors is necessary in the cotangent bundle case, it is automatically satisfied, owing to the Dirichlet condition $\ell(i) \in T_{\ell(i)}(T^*P)$ for paths. In contrast with this, the variational method dealt with in section 2 needs the condition $\Theta_{\gamma(0)}(X(0)) - \Theta_{\gamma(1)}(X(1)) = 0$ in place of the condition $\ell(i) \in T_{\ell(i)}(T^*P)$.

Acknowledgments

The author would like to thank Professor Iwai for careful reading of the manuscript and valuable comments which led to much improvement of this paper. This work has been supported by a JSPS Research Fellowship for Young Scientists, 19-3956.

References

[AM78] Abraham R and Marsden J E 1978 *Foundations of Mechanics* 2nd edn (New York: Benjamin-Cummings)

[Ar89] Arnold V I 1989 *Mathematical Methods of Classical Mechanics (Graduate Texts in Mathematics)* 2nd edn (Berlin: Springer)

[CM87] Cendra H and Marsden J E 1987 Lin constraints, Clebsch potentials and variational principles *Physica* D **27** 63–89

[CIM87] Cendra H, Ibort L A and Marsden J E 1987 Variational principles on principal fiber bundles: a geometric theory of Clebsch potentials and Lin constraints *J. Geom. Phys.* **4** 183–205

[IO96] Ibort A and Ontalba C M 1996 Periodic orbits of Hamiltonian systems and symplectic reduction *J. Phys. A: Math. Gen.* **29** 675–87

[KKS78] Kazhdan D, Kostant B and Sternberg S 1978 Hamiltonian group actions and dynamical systems of Calogero type *Commun. Pure Appl. Math.* **31** 481–507

[MP00] Marsden J E and Perlmutter M 2000 The orbit bundle picture of cotangent bundle reduction *C. R. Math. Acad. Sci. Soc. R. Can.* **22** 35–54

[MW74] Marsden J E and Weinstein A 1974 Reduction of symplectic manifolds with symmetry *Rep. Math. Phys.* **5** 121–30

[OR04] Ortega J-P and Ratiu T S 2004 *Momentum Maps and Hamiltonian Reduction (Progress in Mathematics* vol 222*)* (Basel: Birkhäuser Boston)